UDC 531.314.2

ON A GENERALIZED FORM OF LAGRANGE EQUATIONS OF SECOND KIND

PMM Vol. 37, №1, 1973, pp.156-159

G. DIUKICH
(Yugoslavia)

(Received January 31, 1972)

We have obtained a new form for the equations of motion of both holonomic as well as nonholonomic systems. In a special case these equations reduce to the Mangeron-Deleanu equations [1 – 3].

1. The equations of motion of a rheonomic-holonomic mechanical system with n degrees of freedom, generalized coordinates $q_1, \ldots, q_n$, kinetic energy $T = T(t, q_j, q_j^{\cdot})$, and generalized forces $Q_j = Q_j(t, q_r, q_r^{\cdot})$ can be written in the form [1]

$$\frac{1}{p}\frac{\partial T^{(p)}}{\partial q_j^{(p)}} - \frac{p+1}{p}\frac{\partial T}{\partial q_j} = Q_j \tag{1.1}$$

$$\left(T^{(p)} = \frac{d^p T}{dt^p}, \quad q_j^{(p)} = \frac{d^p q_j}{dt^p}, \quad p = 1, 2 \ldots\right)$$

Here and below $r, j = 1, \ldots, n$, $i = 1, \ldots, m$, $\varkappa, \mu, \nu = 1, \ldots, g$: in what follows summation over repeated subscripts is to be understood. Using an identity from [1], which the kinetic energy satisfies,

$$\frac{\partial T}{\partial q_j} \equiv \frac{1}{p-s}\left[p\frac{\partial T^{(s)}}{\partial q_j^{(s)}} - s\frac{\partial T^{(p)}}{\partial q_j^{(p)}}\right]$$

$$(p \neq s;\ p = 1, 2, \ldots;\ s = 0, 1, 2, \ldots)$$

the equations of motion (1.1) can be written as

$$Z_j(T) = Q_j, \qquad Z_j(T) \equiv \frac{1}{p-s}\left[(1+s)\frac{\partial T^{(p)}}{\partial q_j^{(p)}} - (1+p)\frac{\partial T^{(s)}}{\partial q_j^{(s)}}\right] \tag{1.2}$$

From this generalized form of Lagrange equations of second kind we can obtain: for $s = 0$, the Mangeron-Deleanu equations (1.1), for $s = 0$ and $p = 1$, the Nielsen equations [4], and for $s = 0$ and $p=2$, the Tsenov (Tzénoff) equations [5].

Suppose that the generalized forces have the form

$$Q_j = -\partial V/dq_j - \partial\Phi/dq_j^{\cdot} \tag{1.3}$$

where $V(t, q_j)$ is the potential energy and $\Phi(t, q_j, q_j^{\cdot})$ is the damping function. It can be shown that

$$\frac{\partial V}{\partial q_j} \equiv \frac{\partial V^{(s)}}{\partial q_j^{(s)}}, \qquad \frac{\partial \Phi}{\partial q_j^{\cdot}} \equiv \frac{\partial \Phi^{(p-1)}}{\partial q_j^{(p)}} \qquad \begin{pmatrix} p = 1, 2 \ldots \\ s = 0, 1, 2 \ldots \end{pmatrix}$$

Using these identities and (1.3) the equations of motion (1.2) reduce to

$$Z_j(T) - Q_j \equiv \frac{1}{p-s}\left[(1+s)\frac{\partial L^{(p)}}{\partial q_j^{(p)}} - (1+p)\frac{\partial L^{(s)}}{\partial q_j^{(s)}}\right] = 0 \tag{1.4}$$

$$(p \neq s;\ p = 1, 2 \ldots;\ s = 1, 2, \ldots)$$

where

$$L = T - V - \int_0^t \Phi\, dt \tag{1.5}$$

is a generalized Lagrange function.

2. Let us assume that a mechanical system is nonholonomic and that its motion is constrained by g nonholonomic constraints of the form

$$F_{\mu j}(t, q_r, \ldots, q_r^{(k)})\, q_j^{(k+1)} + F_\mu (t, q_r, \ldots, q_r^{(k)}) = 0 \tag{2.1}$$

where k is an integer ($k \geqslant 1$). These nonholonomic constraints are linear relative to the highest time derivative of the generalized coordinates. Such a mechanical system has $m = n - g$ degrees of freedom. Assume that q_i are the independent generalized coordinates, and $q_{m+\nu}$, the dependent ones. From system (2.1) with g equations we can obtain the relations

$$q_{m+\nu}^{(k+1)} = \alpha_{\nu i}(t, q_j, \ldots, q_j^{(k)})\, q_i^{(k+1)} + \alpha_\nu (t, q_j, \ldots, q_j^{(k)}) \tag{2.2}$$

Using the results of this paper, as well as of [6], we can show that the equations of motion of a mechanical system with g nonholonomic constraints (2.2) of ($k + 1$)-st order have the form

$$Z_i(T) - Q_i = -\lambda_\nu \alpha_{\nu i}$$

$$Z_{m+\nu}(T) - Q_{m+\nu} = \lambda_\nu \tag{2.3}$$

where λ_ν are undetermined Lagrange multipliers and $Z_j(T)$ are determined by the second relation in (1.2). From (2.3) and (1.4) with $s = k + 1$, $p = k + 2$ we obtain

$$(k+2)\left(\frac{\partial L^{(k+2)}}{\partial q_i^{(k+2)}} + \alpha_{\nu i} \frac{\partial L^{(k+2)}}{\partial q_{m+\nu}^{(k+2)}}\right) - (k+3)\left(\frac{\partial L^{(k+1)}}{\partial q_i^{(k+1)}} + \alpha_{\nu i} \frac{\partial L^{(k+1)}}{\partial q_{m+\nu}^{(k+1)}}\right) = 0 \tag{2.4}$$

For the ($k+1$)-st time derivative of the generalized Lagrange function we obtain

$$L^{(k+1)} = L^{(k+1)}(t, q_j, \ldots, q_i^{(k+1)}, q_{m+\nu}^{(k+1)}, q_i^{(k+2)}, q_{m+\nu}^{(k+2)})$$

Eliminating the dependent ($k+1$)-st derivatives $q_{m+\nu}^{(k+1)}$ of the generalized coordinates with the aid of the constraint Eqs. (2.2), we have

$$L_*^{(k+1)}(t, \ldots, q_i^{(k+1)}, q_i^{(k+2)}, q_{m+\nu}^{(k+2)}) = L^{(k+1)}(t, \ldots, q_i^{(k+1)}, q_{m+\nu}^{(k+1)}, q_i^{(k+2)}, q_{m+\nu}^{(k+2)}) \tag{2.5}$$

By a total time differentiation of expression (2.2) we obtain

$$q_{m+\nu}^{(k+2)} = \alpha_{\nu i}(t, q_j, \ldots, q_j^{(k)})\, q_i^{(k+2)} + \alpha_\nu^{*}(t, q_j, \ldots, q_j^{(k+1)}) \tag{2.6}$$

From (2.2), (2.5) and (2.6) follows

$$\frac{\partial L_*^{(k+1)}}{\partial q_i^{(k+1)}} + \left[\frac{\partial L_*^{(k+1)}}{\partial q_{m+\nu}^{(k+2)}} - \frac{\partial L^{(k+1)}}{\partial q_{m+\nu}^{(k+2)}}\right]\left[\frac{\partial q_{m+\nu}^{(k+2)}}{\partial q_i^{(k+1)}} + \alpha_{\varkappa i} \frac{\partial q_{m+\nu}^{(k+2)}}{\partial q_{m+\varkappa}^{(k+1)}}\right] =$$
$$\frac{\partial L^{(k+1)}}{\partial q_i^{(k+1)}} + \alpha_{\nu i} \frac{\partial L^{(k+1)}}{\partial q_{m+\nu}^{(k+1)}} \tag{2.7}$$

From the way in which the function $L_*^{(k+1)}$ was obtained from the function $L^{(k+1)}$ it is clear that

$$\frac{\partial L_*^{(k+1)}}{\partial q_{m+\nu}^{(k+2)}} = \frac{\partial L^{(k+1)}}{\partial q_{m+\nu}^{(k+2)}}$$

Then, with due regard to (2.7), we get

$$\frac{\partial L_*^{(k+1)}}{\partial q_i^{(k+1)}} = \frac{\partial L^{(k+1)}}{\partial q_i^{(k+1)}} + \alpha_{\nu i} \frac{\partial L^{(k+1)}}{\partial q_{m+\nu}^{(k+1)}} \tag{2.8}$$

The following relation is derived analogously:

$$\frac{\partial L_*^{(k+2)}}{\partial q_i^{(k+2)}} = \frac{\partial L^{(k+2)}}{\partial q_i^{(k+2)}} + \alpha_{\nu i} \frac{\partial L^{(k+2)}}{\partial q_{m+\nu}^{(k+2)}} \tag{2.9}$$

where the function $L_*^{(k+2)}$ is obtained by eliminating the dependent derivatives $q_{m+\nu}^{(k+2)}$ in the function $L^{(k+2)}$ by means of constraints (2.6). From (2.8), (2.9) and (2.4) we obtain the generalized form of the Lagrange equations of second kind for mechanical systems with nonholonomic constraints of ($k+1$)-th order ($k \geqslant 1$)

$$(k+2)\frac{\partial L_*^{(k+2)}}{\partial q_i^{(k+2)}} - (k+3)\frac{\partial L_*^{(k+1)}}{\partial q_i^{(k+1)}} = 0 \tag{2.10}$$

These m equations together with the g nonholonomic constraint Eqs. (2.2) form a complete system of n equations for the determination of the n generalized coordinates q_j as functions of time.

3. Example. A material point of mass m moves in a gravitational field (g is the acceleration due to gravity) in the presence of Appell's [7] nonlinear nonholonomic constraint of first order

$$x^{\cdot 2} + y^{\cdot 2} - \frac{z^{\cdot 2}}{a^2} = 0 \qquad (a = \text{const}) \tag{3.1}$$

where x, y, z are the Cartesian coordinates of the material point. By a time differentiation of the first-order nonlinear constraint (3.1) we obtain a second-order quasilinear nonholonomic constraint

$$x^{\cdot}x^{\cdot\cdot} + y^{\cdot}y^{\cdot\cdot} - \frac{z^{\cdot}}{a^2} z^{\cdot\cdot} = 0 \tag{3.2}$$

Here $k = 1$, $\nu = 1$, $n = 3$, the number of degrees of freedom $m = 2$, the Lagrange function is

$$L = \frac{m}{2}(x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}) - mgz \tag{3.3}$$

and the equations of motion (2.10) become

$$3\frac{\partial L_*^{(3)}}{\partial q_i^{(3)}} - 4\frac{\partial L_*^{(2)}}{\partial q_i^{(2)}} = 0 \tag{3.4}$$

Let x and y be the independent coordinates and let z depend on them. By a time differentiation of Eq. (3.2) we obtain

$$z^{(3)} = \frac{a^2}{z^{\cdot}}\left(x^{\cdot}x^{(3)} + y^{\cdot}y^{(3)} + (x^{(2)})^2 + (y^{(2)})^2 - \frac{(z^{(2)})^2}{a^2}\right) \tag{3.5}$$

By a time differentiation of the Lagrange function (3.3) and by substituting for $z^{(2)}$ and $z^{(3)}$ their values from (3.2) and (3.5), we obtain

$$L_*^{(2)} = m\left\{(x^{(2)})^2 + (y^{(2)})^2 + \frac{a^2}{z^{\cdot}}(x^{\cdot}x^{(2)} + y^{\cdot}y^{(2)})\left[\frac{a^2}{z^{\cdot}}(x^{\cdot}x^{(2)} + y^{\cdot}y^{(2)}) - g\right]\right\} + \ldots \tag{3.6}$$

$$L_*^{(3)} = m\{3(x^{(2)}x^{(3)} + y^{(2)}y^{(3)}) + a^2/z^{\cdot}(x^{\cdot}x^{(3)} + y^{\cdot}y^{(3)})(3z^{(2)} - g)\} + \ldots \tag{3.7}$$

Here we have omitted the terms not containing the second derivatives of the generalized coordinates in expression (3.6) and the third derivatives in expression (3.7). From (3.6), (3.7) and (3.4) we obtain the equations of motion in the independent generalized coordinates x and y

$$x^{(2)} + a^2 \frac{x^{\cdot}}{z^{\cdot}} (z^{(2)} + g) = 0, \qquad y^{(2)} + a^2 \frac{y^{\cdot}}{z^{\cdot}} (z^{(2)} + g) = 0$$

They agree with the equations of motion obtained in [8] by another method.

BIBLIOGRAPHY

1. Mangeron, D. and Deleanu, S., Sur une classe d'équations de la méchanique analytique au sens de I. Tzénoff. Dokl. Bolgar. Akad. Nauk, Vol. 15, №1, 1962.
2. Dolaptschiev, Bl., Über die verallgemeinerte Form der Lagrangeschen Gleichungen, welche auch die Behandlung von nichtholonomen mechanischen Systemen gestattet. Z. angew. Math. und Phys., Bd. 17, №3, 1966.
3. Dobronravov, V. V., Fundamentals of the Mechanics of Nonholonomic Systems. Moscow, "Vysshaia Shkola", 1970.
4. Nielsen, J., Vorlesungen über elementare Mechanik. Berlin, Springer-Verlag, 1935.
5. Tsenov, I. (I. Tzénoff), On a new form of the equations of analytical dynamics. Dokl. Akad. Nauk SSSR, Vol. 89, №1, 1953.
6. Maslov, Iu. N., On the most general equations of analytical dynamics. Tr. Tashkent. Univ., №208, 1962.
7. Appell, P. Sur les liaisons exprimées par les relations non lineaires entre les vitesses. C. R. Acad. Sci. Paris, Vol. 152, 1911.
8. Dolaptchiev, Bl., Nouvel example d'application des équations de Nielsen à des systèmes méchaniques non holonomes. C. R. Acad. Sci. Paris, Vol. 267, №12, 1968.

Translated by N. H. C.

UDC 531.36 + 62-50

ON THE STABILITY OF NONLINEAR SYSTEMS WITH A TRANSFORMED ARGUMENT

PMM Vol. 37, №1, 1973, pp. 159-162
V. P. SKRIPNIK
(Moscow)
(Received September 10, 1971)

We study a linear and a perturbed system; in the latter the argument is transformed. Under the assumption that the trivial solution of the linear system is stable, we ascertain the conditions under which the trivial solution of the perturbed system also will be stable.

Let $f(t, \xi_k) = f(t, \xi_1, \xi_2, \ldots \xi_p)$ $(k = 1, 2, \ldots, p)$, where $f, \xi_1, \xi_2, \ldots, \xi_p$ are m-dimensional vectors. We consider the following two mth-order systems: the linear one

$$y' = A(t)\, y \tag{1}$$

and the perturbed one (see [1])